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On Optimum Spines

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This article presents the optimum dimensions and heat transfer characteristics of spines for cylindrical, convex parabolic, conical, and concave parabolic profiles. With a derived solution form, the optimum base diameter, length, heat dissipation, temperature profile, and efficiency of spines are obtained by fin parameter and tip temperature. The temperature-dependent heat transfer coefficient is assumed to be a power-law type. It is found that the optimum dimensions of a spine are functions of fin volume, heat transfer coefficient at fin base, and thermal conductivity. For a fin with least material, the optimum conditions are also obtained. It shows that the results are solely related to the heat transfer mode on the fin surface. To facilitate the thermal design of heat transfer components, simple mathematical expressions as well as design charts are presented.

Nomenclature

a = dimensional constant related to a selected heat transfer mode, W m^{-2} K^{-1}
D = base diameter of a spine, m
f = dimensionless temperature, θ/θ_0
H = constraint function, defined in Eqs. (10), (21), and (30)
h = heat transfer coefficient, W m^{-2} K^{-1}
k = thermal conductivity, W m^{-1} K^{-1}
l = fin length, m
m = power-law exponent
n = fin profile index given in Eq. (2)
Q = heat dissipation of a fin, W
q = surface heat flux, W m^{-2}
V = fin volume, m^3
x = coordinate
ε = emissivity of the fin material
ζ = dimensionless fin length
η = fin efficiency
θ = temperature in excess of the ambient or temperature superheat, $T - T_a$
ξ = dimensionless coordinate
σ = Stefan-Boltzmann constant, W m^{-2} K^{-4}

Subscripts and Superscripts

a = ambient
b = fin base
min = minimum
opt = optimum
* = nondimensional quantity
0 = fin tip

Introduction

THE use of fins is an effective method to increase the heat dissipation from a surface. Applications for finned surfaces are widely seen in air-conditioning, aircraft, chemical processing plants, cryogenics and refrigeration, and the cooling of electronic components. Since weight and material costs are the primary design considerations in most of these applications, it is highly desirable to obtain the optimum design information of fins.

The optimum design of extended surfaces is available in the work of Kern and Kraus.[1] Lately, a thorough review for finned surface technology has been presented by Kraus.[2] For spines, Sonn and Bar-Cohen[3] and Li[4] dealt with the optimization of a single pin fin and fin array, respectively. Considering convective spines of cylindrical, convex parabolic, conical, and concave parabolic profiles with curvature effect, Chung et al.[5] have presented a systematic study on the optimum dimensions for these fins. As for radiative heat transfer, Chung and Nguyen[6] formulated the general relationships for spine dimensions as well as heat transfer characteristics under the optimum condition. In addition, the optimum geometry of the fin with the least weight is uniquely determined in their work.

For fins in boiling heat transfer, the heat transfer coefficient can be expressed as an empirical function of the temperature difference between the fin surface and liquid. In the fin optimization problems, Haley and Westwater[7] used numerical computations to find a turnip-shaped spine that dissipates heat at the smallest volume. Yeh and Liaw[8] proposed the optimum excavated shape in a cylindrical fin. Employing a temperature-correlated profile, Razani and Zohoor[9] and Sohrabpour and Razani[10] investigated the optimization of fins with temperature-dependent heat transfer coefficient. Very recently, using an integral approach, Chung and Iyer[11] have determined the optimum dimensions for longitudinal rectangular fins and cylindrical pin fins by incorporating transverse heat conduction. Due to the complexity of their results, only numerous design charts are given.

In this study, the dimensions and heat transfer characteristics of an optimum spine in boiling as well as natural convective heat transfer are investigated. For given cylindrical, convex parabolic, conical, and concave parabolic profiles, the base diameter and length of a spine that maximize the heat duty for a fixed volume are initially obtained. Considering a power-law-type heat transfer coefficient, simple formulas for the base diameter, length, heat dissipation, temperature distributions, and efficiency of the optimum spines are derived. In addition, the least-material fin profile of a spine as well as the corresponding optimum data are evaluated for various heat transfer modes.

Analysis

Optimum Dimensions of Spines

A spine with an arbitrary profile is now considered. The surface heat flux along the fin length varies following a power-law-type dependence on the temperature difference between the fin and the ambient fluid, i.e.,

$$q = a\theta^m \tag{1}$$

where a and m are dimensional and dimensionless constants,

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respectively. The values of a and m depend on the boiling liquids and the types of heat transfer. For instance, the exponent m may take the values of 0.75, 1, 1.25, 1.33, 3, and 4, depending on whether the heat transfer process dominates by film boiling, forced convection, laminar-free convection, turbulent-free convection, nucleate boiling, or radiation into free space at zero temperature. In forced convection, a is equivalent to the constant heat transfer coefficient h. For $m = 4$, a stands for the product of emissivity ε, of the fin material and Stefan-Boltzmann constant σ.

The following assumptions are made through these analyses:

1) The material of the fin is homogeneous and the thermal conductivity of the fin is constant.
2) No heat source or sink exists within the fin.
3) The ambient fluid temperature is maintained at a constant value T_a.
4) There is no bond resistance at the base and the base temperature T_b is uniform.
5) One-dimensional heat conduction along the spine, steady state, and length of arc simplification are assumed. The general fin profile function can be expressed by

$$f_1(x) = (D/2)(x/l)^n \tag{2}$$

where D and l denote the base diameter and length of a fin, respectively, and the profile index represents cylindrical, convex parabolic, conical, and concave parabolic profiles for $n = 0, 0.5, 1$, and 2, respectively. The volume of spines may be written in terms of base diameter, length, and profile index, and is given by $V = \pi D^2 l/[4(2n + 1)]$. Applying the aforementioned assumptions and substituting the general fin profile yields

$$\frac{d}{dx}\left(x^{2n}\cdot\frac{d\theta}{dx}\right) - \frac{4a}{kD}(lx)^n\cdot\theta^m = 0 \tag{3}$$

where k is thermal conductivity of a fin. The terminology and coordinate system are shown in Fig. 1. Note that the origin of the coordinate system is taken at fin tip, and positive x is toward the fin base.

Spine of Cylindrical Profile

The governing equation for this case is given by

$$\frac{d^2\theta}{dx^2} - \frac{2a}{kD}\cdot\theta^m = 0 \tag{4}$$

and the associated boundary conditions are

$$\frac{d\theta}{dx}(0) = 0 \tag{5a}$$

$$\theta(l) = \theta_b \tag{5b}$$

After some substitutions and manipulations, Eq. (4) can be written as

$$f''(\xi) - [f(\xi)]^m = 0 \tag{6}$$

where

$$f(\xi) = (\theta/\theta_0) \tag{7a}$$

$$\xi = (2a\theta_0^{m-1}/kD)^{1/2}\cdot x = \zeta\cdot\theta_0^{(m-1)/2} \tag{7b}$$

In the above equations, θ_0 is the temperature at fin tip. The heat flow through the fin base can be evaluated from

$$Q = (\pi/2)(akD^3)^{1/2}\cdot\theta_0^{(m+1)/2}\cdot f'[\zeta_b\theta_0^{(m-1)/2}] \tag{8}$$

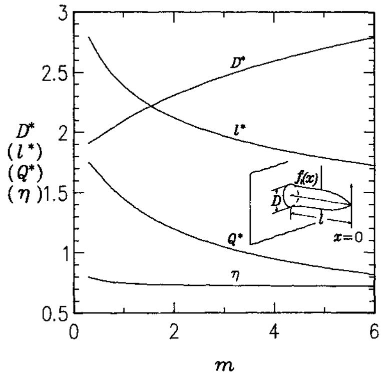

Fig. 1 Dimensionless optimum base diameter, length, heat duty, and efficiency for a spine of concave parabolic profile.

where ζ_b may be referred as fin parameter. It is desired to maximize Q by varying ζ_b. However, θ_0 varies with ζ_b. Thus, Q is, in general, a function of ζ_b and θ_0. Consider that Q and H are functions of ζ_b and θ_0 for a given physical condition and fixed fin volume, $V = \pi D^2/4$, Eqs. (7) and (8) may be rewritten in the forms:

$$Q(\zeta_b, \theta_0) = (16\pi^2a^4kV^3)^{1/5}\cdot\zeta_b^{-3/5}\cdot\theta_0^{(m+1)/2}\cdot f'[\zeta_b\theta_b^{(m-1)/2}] \tag{9}$$

$$H(\zeta_b, \theta_0) = \theta_b - \theta_0\cdot f[\zeta_b\theta_0^{(m-1)/2}] = 0 \tag{10}$$

Note that $H(\zeta_b, \theta_0) = 0$ is a constraint equation. The equivalent problem of optimizing Q will be to find the extreme values of function $Q(\zeta_b, \theta_0)$ subject to the subsidiary constraint condition $H(\zeta_b, \theta_0) = 0$. The stationary point thus found gives the desired optimum values of ζ_b and θ_0. The solution to this problem can be obtained by Lagrange's multiplier method. Eliminating the Lagrange's multiplier gives

$$\left(\frac{\partial Q}{\partial\zeta_b}\right)\cdot\left(\frac{\partial H}{\partial\theta_0}\right) - \left(\frac{\partial H}{\partial\zeta_b}\right)\cdot\left(\frac{\partial Q}{\partial\theta_0}\right) = 0 \tag{11}$$

From Eqs. (10–12), a relationship of optimum ξ_b is derived as

$$(4m + 1)\cdot\xi_b\cdot[f'(\xi_b)]^2 + 3f(\xi_b)f'(\xi_b) - 5\xi_b\cdot[f(\xi_b)]^{m+1} = 0 \tag{12}$$

The fourth-order Runge-Kutta method is used to solve Eq. (6) for which the starting values: $f(0) = 1$, $f'(0) = 0$, and $f''(0) = 1$ are easily obtained. The equation is integrated out to the optimum fin length at which Eq. (12) is satisfied. The dimensionless temperature distribution of the optimum fin is calculated from

$$\frac{\theta}{\theta_b} = \frac{f(\xi)}{f(\xi_b)} \tag{13}$$

The dimensionless optimum base diameter and length of the fin become

$$D^* = \frac{D_{opt}}{(h_bV^2/k)^{1/5}} = (64/\pi^2)^{1/5}\cdot\xi_b^{2/5}\cdot[f(\xi_b)]^{-(m-1)/5} \tag{14}$$

$$l^* = \frac{l_{opt}}{(k^2V/h_b^2)^{1/5}} = [1/(4\pi)]^{1/5}\cdot\xi_b^{4/5}\cdot[f(\xi_b)]^{2(m-1)/5} \tag{15}$$

and the maximum heat dissipation is obtained as

$$Q^* = \frac{Q_{opt}}{\theta_b(h_b^4kV^3)^{1/5}} = (16\pi^2)^{1/5}\cdot\xi_b^{-1}\cdot f'(\xi_b)\cdot[f(\xi_b)]^{-(4m+1)/5} \tag{16}$$

In the above equations h_b ($=a\theta_b^{m-1}$) is the heat transfer coefficient at fin base. The spine efficiency is defined as the ratio of the actual heat dissipated by the fin to the ideal heat transferred if the entire spine is operating at the base temperature. Hence, the efficiency of the optimum fin can be expressed as

$$\eta = \frac{f'(\xi_b)}{\xi_b \cdot [f(\xi_b)]^m} \tag{17}$$

Spine of Convex Parabolic Profile

In this case, the boundary conditions are the same as that given in Eqs. (5a) and (5b). Employing the same $f(\xi)$ as defined in Eq. (7), the governing equation can be rewritten as

$$\xi \cdot f''(\xi) + f'(\xi) - \xi^{1/2} \cdot [f(\xi)]^m = 0 \tag{18}$$

where the dimensionless parameter ξ is given by

$$\xi = \left(\frac{4al^{1/2}}{kD}\right)^{2/3} \cdot x \cdot \theta_0^{2(m-1)/3} = \zeta \cdot \theta_0^{2(m-1)/3} \tag{19}$$

For a fixed fin volume, Q and H functions are derived in the following forms:

$$Q(\zeta_b, \theta_0) = (128a^4kV^3)^{1/5} \cdot \zeta_b^{-1/5} \cdot \theta_0^{(2m+1)/3} \cdot f'[\zeta_b \theta_0^{2(m-1)/3}] \tag{20}$$

$$H(\zeta_b, \theta_0) = \theta_b - \theta_0 \cdot f[\zeta_b \theta_0^{2(m-1)/3}] = 0 \tag{21}$$

and an equation at the optimum condition is obtained as

$$(4m+1) \cdot \xi_b \cdot [f'(\xi_b)]^2 + 6f(\xi_b)f'(\xi_b) - 5\xi_b^{1/2} \cdot [f(\xi_b)]^{m+1} = 0 \tag{22}$$

Also, the fourth-order Runge-Kutta method is used to solve Eq. (18) for which $f(0) = 1$, $f'(0) = 0$, and $f''(0) = 0$. Note that the value of $f''(0)$ is determined by applying l'Hospital's rule. The dimensionless optimum base diameter, length, heat dissipation, and efficiency of the fin are the same as those defined in Eqs. (14–17), and are evaluated from the following forms:

$$D^* = (256/\pi^2)^{1/5} \cdot \xi_b^{-3/10} \cdot [f(\xi_b)]^{-(m-1)/5} \tag{23}$$

$$l^* = [1/(2\pi)]^{1/5} \cdot \xi_b^{3/5} \cdot [f(\xi_b)]^{2(m-1)/5} \tag{24}$$

$$Q^* = (128\pi^2)^{1/5} \cdot \xi_b^{-1/5} \cdot f'(\xi_b) \cdot [f(\xi_b)]^{-(4m+1)/5} \tag{25}$$

$$\eta = \frac{3f'(\xi_b)}{2\xi_b^{1/2} \cdot [f(\xi_b)]^m} \tag{26}$$

Spine of Conical Profile

The governing equation may be derived in the form

$$\xi \cdot f''(\xi) + 2f'(\xi) - [f(\xi)]^m = 0 \tag{27}$$

where

$$\xi = (4al/kD) \cdot x \cdot \theta_0^{m-1} = \zeta \cdot \theta_0^{m-1} \tag{28}$$

Similar to previous cases, Q and H functions are expressed as

$$Q(\zeta_b, \theta_0) = (432\pi^2a^4kV^3)^{1/5} \cdot \zeta_b^{1/5} \cdot \theta_0^m \cdot f'(\zeta_b \theta_0^{m-1}) \tag{29}$$

$$H(\zeta_b, \theta_0) = \theta_b - \theta_0 \cdot f(\zeta_b \theta_0^{m-1}) = 0 \tag{30}$$

An optimum constraint at fin base is derived as

$$(4m+1) \cdot \xi_b \cdot [f'(\xi_b)]^2 + 9f(\xi_b)f'(\xi_b) - 5[f(\xi_b)]^{m+1} = 0 \tag{31}$$

The initial values can be determined in view of Eq. (27). It turns out that $f(0) = 1$ and $f'(0) = 1$. In addition, $f''(0)$ is equal to $m/2$ by using l'Hospital's rule. The dimensionless optimum dimensions, heat dissipation, and efficiency of a fin becomes

$$D^* = (576/\pi^2)^{1/5} \cdot \xi_b^{-1/5} \cdot [f(\xi_b)]^{-(m-1)/5} \tag{32}$$

$$l^* = [3/(4\pi)]^{1/5} \cdot \xi_b^{2/5} \cdot [f(\xi_b)]^{2(m-1)/5} \tag{33}$$

$$Q^* = (432\pi^2)^{1/5} \cdot \xi_b^{1/5} \cdot f'(\xi_b) \cdot [f(\xi_b)]^{-(4m+1)/5} \tag{34}$$

$$\eta = \frac{2f'(\xi_b)}{[f(\xi_b)]^m} \tag{35}$$

In view of Eq. (3), the solution of a fin of concave parabolic profile exists only when $\theta_0 = 0$. Thus, the optimization by means of ζ_b and θ_0 cannot be used here. Nevertheless, the optimum dimensions of the fin for $n = 2$ can be obtained by the method suggested by Kern and Kraus[1] or Chung and Nguyen.[6]

Fins of Least Material

Following and modifying the approach suggested by Chung and Nguyen[6] or Wilkins,[12] one obtains the fin profile function in the form:

$$f_1(x) = \frac{2Q_b^2(4m+1)}{5\pi^2 h_b \theta_b^2 k} \cdot \left(\frac{x}{l}\right)^{(9m+1)/(2m+3)} \tag{36}$$

Comparing the above equation with Eq. (2), it is concluded that $n = (9m + 1)/(2m + 3)$ yields the design of least material profile for all m. Furthermore, the corresponding base diameter, length, heat dissipation, and efficiency of the fin are derived as follows:

$$D^* = \left[\frac{80(4m+1)}{\pi^2}\right]^{1/5} \tag{37}$$

$$l^* = \frac{1}{(2m+3)} \cdot \left[\frac{500(4m+1)^3}{\pi}\right]^{1/5} \tag{38}$$

$$Q^* = \left[\frac{625\pi^2}{16(4m+1)}\right]^{1/5} \tag{39}$$

$$\eta = \frac{11m+4}{4(4m+1)} \tag{40}$$

Results and Discussion

For spines with cylindrical, convex parabolic, conical, and concave parabolic profiles, the optimum base diameter, length, temperature distribution, heat duty, and efficiency of the fins are obtained for many heat transfer modes. It is observed that fin parameter and tip temperature are main factors in the proposed method of fin optimization problem for boiling heat transfer. The optimum dimensions and heat transfer characteristics of a fin totally depend on fin volume, heat transfer coefficient at fin base, and thermal conductivity of a spine for a given fin profile. The calculated parameters are listed in Table 1 for $n = 0$, 0.5, and 1. The optimum data of these fins may be directly evaluated from ξ_b, $f(\xi_b)$, and $f'(\xi_b)$. To verify the present model, a comparison is made between this study and previous works. It is shown that all the data are identical to the results of Sonn and Bar-Cohen[3] and Chung et al.[5] for forced convection, and Chung and Nguyen[6] for radiative heat transfer.

In boiling heat transfer, the values of m may vary from 2 to 5.5, as indicated in the works of Dhir and Liaw[13] and

Table 1 Optimum parameters of cylindrical pin fins, convex parabolic fins, and conical fins for m = 0.75, 1, 1.25, 1.33, 3, and 4

	$n = 0$			$n = 0.5$			$n = 1$		
m	ξ_b	$f(\xi_b)$	$f'(\xi_b)$	ξ_b	$f(\xi_b)$	$f'(\xi_b)$	ξ_b	$f(\xi_b)$	$f'(\xi_b)$
0.75	1.621	1.233	1.076	2.026	1.123	1.574	3.402	0.979	3.297
1	1.453	1.054	0.919	1.714	0.995	1.240	2.444	0.943	2.051
1.25	1.356	0.936	0.815	1.547	0.910	1.039	2.025	0.925	1.483
1.33	1.334	0.906	0.786	1.509	0.888	0.991	1.937	0.921	1.361
3	1.143	0.593	0.516	1.206	0.657	0.544	1.333	0.889	0.500
4	1.106	0.512	0.445	1.152	0.593	0.444	1.240	0.884	0.362

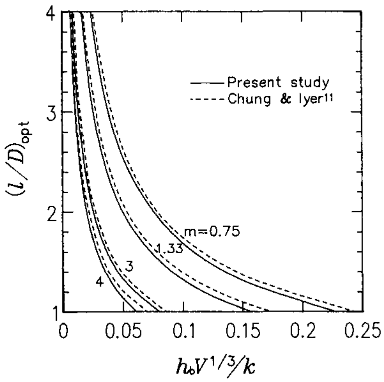

Fig. 2 Comparison of optimum dimensions ratio of this study and two-dimensional analysis.

Gaertner[14] for nucleate boiling, and $0 < m < 1$ for film boiling[7] or film condensation. Figure 1 displays the base diameter, length, heat duty, and efficiency of an optimum concave parabolic spine for $0 \le m \le 6$. As a whole, fin efficiency decreases abruptly with increasing m for $m < 1$, and changes slightly for $2 \le m$. For a conical and a concave-parabolic-profile fin, it is found that no optimum design is found for m near zero. This is due to the fact that the dimensionless temperature drops to zero on the portion near the fin tip for $0 < m < 1$. This result is similar to the finding of Yeh and Liaw.[15] Since inefficient portions begin to show up on the free end of a fin, the optimum conditions of the spine do not exist.

Figure 2 shows the comparison of optimum dimensions ratio of this work and the two-dimensional analysis.[11] It is observed that the deviation is large for a smaller aspect ratio, however, the difference becomes insignificant with an increasing aspect ratio. For a fin with least material, the optimum data are purely related to the power-law exponent m, i.e., the dominant heat transfer mode on the fin surface.

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